WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 5th Semester Examination, 2022-23

MTMACOR12T-MATHEMATICS (CC12)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Show that the function $f:\mathbb{R}^+\to\mathbb{R}^+$ defined by $f(x)=\sqrt{x}$ for all $x\in\mathbb{R}^+$ is an automorphism of the multiplicative group of positive real numbers.

(b) Consider the elements $a=(1\ 2\ 3)$ and $b=(1\ 4)$ in S_4. Determine the commutator $[a, b]$ of a and b in S_4.

(c) Let $X=\{1, 2, 3, 4, 5\}$ and suppose that G is the permutation group defined as $\{(1), (1\ 2\ 3), (1\ 3\ 2), (4\ 5), (1\ 2\ 3)(4\ 5), (1\ 3\ 2)(4\ 5)\}$. Let X be the G-set under the action $\sigma.x=\sigma(x)$, for all $\sigma\in G$, $x\in X$. Find all the distinct orbits of X under the given action.

(d) Is there any group of order 9 whose class equation is given by $9 = 1+1+1+3+3$? Justify your answer.

(e) Show that $Z(G)$ is a characteristic subgroup of G.

(f) Let G be a group of order 125 then show that G has a non-trivial Abelian subgroup.

(g) Prove or disprove: Every group of order 76 contains a unique element of order 19.

(h) Prove that the external direct product $\mathbb{Z}_2\times\mathbb{Z}_3$ of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ is isomorphic with the group $\mathbb{Z}_6$.

(i) Prove or disprove: A_4 is simple.

2. (a) Let G denote the Klein's 4-group. Find the order of the automorphism group $\mathrm{Aut}(G)$ of G. 2

(b) Let G be a group and for each $a\in G$, $f_a : G\to G$ denote the mapping defined by $f_a(g)=gag^{-1}$ for all $g\in G$. Consider the set $\mathrm{Inn}(G)=\{f_a : a\in G\}$. Prove that $\mathrm{Inn}(G)$ is a normal subgroup of the automorphism group of G. 4

(c) Give examples of two non-isomorphic finite groups whose automorphism groups are isomorphic to each other. Justify your choice of groups. 2

3. (a) Show that commutator subgroup of a group G is a characteristic subgroup of G. 3

(b) Show that every characteristic subgroup is a normal subgroup but the converse need not be true. 3

(c) Let $U(n)$ denote the group of units modulo $n>1$. Express $U(144)$ as an external direct product of cyclic groups. 2

4. (a) Show that the group of all automorphisms of a finite cyclic group of order n is isomorphic to the group U_n of units modulo n. 4

(b) Determine the group of all automorphisms of the additive group of all multiples of 3. 4

5. (a) If G be a cyclic group of order mn where $\text{g.c.d}(m, n) = 1$ show that G is isomorphic to the external direct product $P \times Q$ where order of the group P is m and order of the group Q is n. 4

(b) Determine the number of elements of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$, the external direct product of the groups $\mathbb{Z}_{25}$ and $\mathbb{Z}_5$. 4

6. (a) State fundamental theorem of finite abelian groups. 2

(b) Describe all the abelian groups of order 539. Hence show that every such abelian group has an element of order 77. 4+2

7. (a) Let G be a finite group of order 847 and H be a subgroup of G of index 7. Apply generalized Cayley's theorem to show that H is a normal subgroup of G. 4

(b) Find the number of distinct conjugacy classes of the symmetric group S_5. Determine the order of the conjugacy class of the permutation $\alpha = (1\ 2)(3\ 4)$ in S_5. 1+3

8. (a) Let G be a group of permutations of a set S. For each $s \in S$ define stabilizer of S in G and orbit of s under G. 1+1+4
Show that, for any finite group of permutations of a set S,
$$|G| = |\text{orb}_G(s)|\,|\text{stab}_G(s)| \quad \forall s \in S.$$

(b) Let $G = \{(1),\ (1\ 2\ 3)(4\ 5\ 6)(7\ 8),\ (1\ 2\ 3)(4\ 5\ 6)(1\ 3\ 2)(4\ 6\ 5),\ (1\ 3\ 2)(4\ 6\ 5)(7\ 8)\}$ 1+1
Find $\text{orb}_G(4)$ and $\text{stab}_G(4)$.

9. (a) Let G be a finite group of order $p^n m$, where p is a prime integer, n is a non-negative integer and m is a positive integer such that p does not divide m. If n_p denotes the number of Sylow p-subgroups of G, prove the following assertions: 3+2
(i) $n_p \equiv 1 \pmod p$, (ii) n_p devides $|G|$.

(b) Let G be a group of order 99. If G has a normal subgroup of order 9, show that G is a commutative group. 3

10. (a) Let G_1 and G_2 be two groups. Prove that the direct product $G_1 \times G_2$ is commutative if and only if both G_1 and G_2 are commutative. 2

(b) Show that the direct product $Z_6 \times Z_4$ of the cyclic groups Z_6 and Z_4 is not a cyclic group. 3

(c) Find all Abelian groups of order 63 which contain an element of order 21. 3

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